

Restoration of Macroscopic Isotropy on $(d + 1)$ -Simplex Fractal Conductor Networks

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Abstract

Restoration of macroscopic isotropy has been investigated in $(\mathbf{d} + \mathbf{1})$ -simplex fractal conductor networks via exact real space renormalization group transformations. Using some theorems of fixed point theory, it has been shown very rigorously that the macroscopic conductivity becomes isotropic for large scales and anisotropy vanishes with a scaling exponent which is computed exactly for arbitrary values of \mathbf{d} and decimation numbers $\mathbf{b} = \mathbf{2}, \mathbf{3}, \mathbf{4}$ and $\mathbf{5}$.

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1 INTRODUCTION

Restoration of isotropy in an anisotropic system is of great interest in a variety of disciplines where much attention has been focused on it, particularly on the problem of diffusion in inhomogeneous materials [1, 2, 3]. In general, diffusion on lattices can be formulated in terms of an **AC** electric problem and **DC** electric response in a percolating structures can be viewed as a very special case of diffusion in disordered medium [3, 4, 5]. The purpose of this paper is to investigate the restoration of macroscopic isotropy in $(\mathbf{d} + 1)$ -simplex fractal conductor networks with microscopic anisotropy. In general, deterministic fractal lattices[6, 7, 8, 9], as proposed by Kirkpatrick, mimic some properties of percolation clusters in random media and disordered systems[8], and among fractal objects, the $(\mathbf{d} + 1)$ -simplex fractal is the simplest one to study various physical problems from random walk [10, 11, 12, 13] to electrical problem on it[14, 16]. Using the exact renormalization technique based on the minimization of total dissipative power (**TDP**) in these networks, we present a rigorous proof that the conductivity becomes isotropic for large scales, and anisotropy vanishes with a scaling exponent $\bar{\lambda}$, as $\mathbf{L}^{-\bar{\lambda}}$ [19]. We exactly compute $\bar{\lambda}$ for arbitrary values of \mathbf{d} and decimation numbers $\mathbf{b} = 2, 3, 4$ and 5. The contents of this paper is as follows:

Section II presents a brief description of $(\mathbf{d} + 1)$ -simplex fractals with decimation number \mathbf{b} together with an explanation of labelling their subfractal and vertices with the partitions of positive integers[17, 18], where this coding plays a very important role throughout the article. In section III we consider the most general network that can be built in a deterministic way, by putting circuit elements on the bonds of $(\mathbf{d} + 1)$ -simplex fractal of a given generation \mathbf{n} with decimation number \mathbf{b} . In order for the self-similarity of the structure to be preserved in the presence of anisotropy at microscopic level, the nature of the circuit elements, namely its resistances, must depend on the orientation of the bonds. It is clear that in $(\mathbf{d} + 1)$ -simplex there are $\frac{\mathbf{d}(\mathbf{d}+1)}{2}$ different orientations. Then we try to establish recursion equations for the connection resistances which represent the conductivity of these networks, on two successive length scales \mathbf{L} and $\mathbf{L}' = \mathbf{bL}$. In general, these recursion relations are very involved. Fortunately, we do not need to have the explicit form of these recurrence equations, for the investigation of the restoration of isotropy. All we need here is the general properties of these maps, which can be obtained through some physical requirements and assumptions. It should be stressed that these circuits are not fictitious, since $(\mathbf{d} + 1)$ -simplex fractals are embedible in Euclidean 2-dimensions, hence they can be considered as two-dimensional networks, see Fig. 1. Section IV is devoted to a very rigorous proof of the uniqueness of the fixed point of the real

space renormalization group transformation of the ratios of the connection resistances. Here in this section we show that all flows of the real space renormalization group transformation of the connection resistances, stemming from the finite physical region of connection resistance space, diverges to a direction which makes equal angle with all coordinates axes. The proof is based on some theorems and definitions of fixed point theory of the maps on complete metric spaces with the Hilbert metric. We have quoted the required theorems without presenting their proofs, since this section would be otherwise more mathematical in style. We refer the readers to reference [24] for proofs of all theorems and for more details. Those readers who are only interested in the results of this section can skip it. In section V by minimizing the **TDP** in isotropic state, we get linear equations for the inner inward flowing currents in terms of input currents with Lagrange multipliers as their coefficients. Then using $\mathbf{S}_{(\mathbf{d}+1)}$ symmetry group of the $(\mathbf{d} + 1)$ -simplex, we suggest an ansatz for the Lagrange multipliers which leads to determination of the inner flowing currents in terms of the input one for any values of \mathbf{d} and decimation number $\mathbf{b} = 2, 3, 4$ and 5 . Section VI contains the main results of the article. Here in this section, by linearising the recurrence relation of the connection resistances near the isotropy state, we calculate power scaling exponent and the scaling exponent of the suppression of the anisotropy, for arbitrary values of \mathbf{d} and decimation numbers $\mathbf{b} = 2, 3, 4$ and $\mathbf{b} = 5$, which are in agreement with the results of references [4, 19, 20, 21, 22, 23] in special cases. Also these results hold true for $(\mathbf{d} + 1)$ -honeycomb fractal conductor network with decimation number $\mathbf{b} = 2$, see Fig. 2, which is in agreement with reference [16] for $\mathbf{d} = 2$ and $\mathbf{b} = 2$ case. The paper ends with a brief conclusion.

2 $(\mathbf{d}+1)$ -Simplex Fractals

$(\mathbf{d}+1)$ -simplex fractal is a generalization of a two dimensional Sierpinski gasket to \mathbf{d} -dimensions such that its subfractals are $(\mathbf{d}+1)$ -simplices or \mathbf{d} -dimensional polyhedra with $\mathbf{S}_{(\mathbf{d}+1)}$ -symmetry. In order to obtain a fractal with decimation number \mathbf{b} , we choose a $(\mathbf{d} + 1)$ -simplex and divide all the links (that is the lines connecting sites) into \mathbf{b} parts and then draw all possible \mathbf{d} -dimensional hyperplanes through the links parallel to the transverse \mathbf{d} -simplices. Next, having omitted every other innerpolyhedra, we repeat this process for the remaining simplices or for the subfractals of next higher generation. This way through $(\mathbf{d} + 1)$ -simplex fractals are constructed. In order to calculate the fractal dimension, also to determine the current

distribution, it is convenient to label subfractals of generation $(n+1)$ in terms of partition of $(\mathbf{b} - \mathbf{1})$ into $(\mathbf{d} + \mathbf{1})$ positive integers $\lambda_1, \lambda_2, \dots, \lambda_{\mathbf{d}+1}$. Each partition represents a subfractal of generation \mathbf{n} , and λ shows the distance of the corresponding subfractal from \mathbf{d} -dimensional hyper-planes which construct the $(\mathbf{d} + \mathbf{1})$ simplex. On the other hand, each vertex denoted by partition of \mathbf{b} into $(\mathbf{d} + \mathbf{1})$ non-negative integers $\eta_1, \eta_2, \dots, \eta_{\mathbf{d}+1}$ and obviously the \mathbf{i} -th vertex of subfractal $(\lambda_1, \lambda_2, \dots, \lambda_{\mathbf{d}+1})$ is denoted by $\eta_{\mathbf{j}} = \lambda_{\mathbf{j}} + \delta_{\mathbf{i},\mathbf{j}}$, where $\mathbf{j} = \mathbf{1}, \mathbf{2}, \dots, \mathbf{d} + \mathbf{1}$. As an illustrating example we show in Fig. 3 the method of labelling a Sierpinski gasket with decimation number $\mathbf{b} = \mathbf{3}$.

Obviously the number of all possible partitions is equal to the distribution of $(\mathbf{b} - \mathbf{1})$ objects amongst $(\mathbf{d} + \mathbf{1})$ boxes, which is the same as the Bose-Einstein distribution of $(\mathbf{b} - \mathbf{1})$ identical bosons in $(\mathbf{d} + \mathbf{1})$ quantum states. This is equal to

$$C = \frac{(b + d - 1)!}{(b - 1)!d!}. \quad (2-1)$$

As is well known, the fractal dimension \mathbf{D}_f of a self similar object is defined according to [6]

$$NL_f^D = 1$$

where \mathbf{N} is the number of similar objects, up to translation and rotation, here being equal to the number of subfractals of generation \mathbf{n} , and \mathbf{L} is the scale of subfractal of generation \mathbf{n} .

Hence

$$N = C^r, L = b^{-r}$$

Therefore,

$$D_f = \frac{\ln c}{\ln b},$$

or

$$D_f = \frac{\ln\left(\frac{(b+d-1)!}{(b-1)!}\right)}{\ln b}. \quad (2-2)$$

3 Fractal Connection Resistances and their Exact Renormalization Group Transformations

A two-dimensional anisotropic $(\mathbf{d} + \mathbf{1})$ -simplex resistor network consists of $(\mathbf{d} + \mathbf{1})$ nodes, with \mathbf{I}_i denoting the amount of current injected into the network through the node \mathbf{i} and $\frac{\mathbf{d}(\mathbf{d}+1)}{2}$

different resistors (coated with insulator) mutually connecting all the nodes of the network (see Fig (2)).

As usual, total dissipative power **TDP** in these networks can be written in terms of the resistances and the currents flowing in them. But it is more convenient and also advantageous throughout this article to express **TDP** in terms of the inward flowing currents $\mathbf{I}_i, i = 1, 2, \dots, d+1$. In that case, it is clear that **TDP** is a bilinear function of the input currents with the coefficients which have the dimensions of the resistance.

Hence we call these coefficients, connection resistances denoted by $R_{jk}, j, k = 1, 2, \dots, d+1$. Therefore, **TDP** of the network assumes the following form

$$\mathbf{TDP}(\text{network}) = \sum_{j,k=1}^{(d+1)} R_{jk} I_j I_k. \quad (3-1)$$

It is clear from equation (3-1) that \mathbf{R}_{jk} is symmetric with respect to the interchange of indices \mathbf{i} and \mathbf{j} . Also the diagonal elements $\mathbf{R}_{jj}, j = 1, 2, \dots, d+1$ can be eliminated from the expression (3-1), if we use Kirchhoff's current law for the input currents

$$\sum_{j=1}^{(d+1)} I_j = 0. \quad (3-2)$$

Thus, The expression (3-1) takes the following form

$$\mathbf{TDP}(\text{network}) = - \sum_{j \neq k=1}^{(d+1)} R_{jk} I_j I_k = -2 \sum_{k>j=1}^{(d+1)} R_{jk} I_j I_k. \quad (3-3)$$

From positive definiteness of **TDP** for all arbitrary values of input inward flowing currents consistent with Kichhoff's current law, it follows that all connection resistances are positive, that is we have:

$$R_{jk} > 0 \quad \text{for all } k > j = 1, 2, \dots, d+1.$$

From the form of the **TDP** given in (3-3), it also follows that there is a bijective map between these sets of the independent connection resistances $\{\mathbf{R}_{jk}, k > j = 1, 2, \dots, d+1\}$ and $\frac{d(d+1)}{2}$ mutual resistors of $(d+1)$ -simplex network. Accordingly, these independent connection resistances can represent the mutual resistors of the network and in the case of an anisotropic network the connection resistances will be different. Consequently, for the investigation of the restoration of macroscopic isotropy in $(d+1)$ -simplex fractal resistor lattices, by real space renormalization group method, we need to know the recursion relations between the connection resistances of a given generation and the connection resistances of one generation below it.

These recursion relations can easily be obtained if we compare the total dissipative power **TDP** of generation \mathbf{n} given in (3-3) with the same quantity, calculated as sum of power of its $(\mathbf{n} - 1)$ th generated subfractals which can be expressed as a function of connection resistances of generation $\mathbf{n} - 1$, provided that in calculating the power of its subfractals, the inner inward flowing currents are stated in terms of input currents. To determine these currents it is convenient to denote the \mathbf{j} -th inward flowing current of subfractal corresponding to the partition $\lambda_1, \lambda_2, \dots, \lambda_{\mathbf{d}+1}$ by $\mathbf{I}_{\lambda_1, \lambda_2, \dots, \lambda_{\mathbf{d}+1}}(\lambda_1, \dots, \lambda_{\mathbf{j}-1}, \lambda_{\mathbf{j}}+1, \lambda_{\mathbf{j}+1}, \dots, \lambda_{\mathbf{d}+1})$. Thence $\mathbf{I}_{\mathbf{j}}$, the \mathbf{j} -th inward flowing current of $(\mathbf{d} + 1)$ -simplex fractal, is given by

$$I_{0,0,\dots,0,\underbrace{1}_{j\text{-th}},0,\dots,0}(0,0,\dots,0,\underbrace{2}_{j\text{-th}},0,\dots,0) = I_j.$$

To determine the inner inward flowing currents, besides applying Kirchhoff's current law at each node and subfractal, we have to minimize the total dissipative power of $(\mathbf{d} + 1)$ -simplex fractal of generation \mathbf{n} , calculated as the sum of the **TDP** of its subfractals as:

$$\begin{aligned} & \sum_{\substack{j,k=1 \\ \text{sum over partition of } (b-1)}}^{d+1} R_{jk}(n-1) I_{\lambda_1, \dots, \lambda_{d+1}}(\lambda_1, \dots, \lambda_j + 1, \dots, \lambda_{d+1}) I_{\lambda_1, \dots, \lambda_{d+1}}(\lambda_1, \dots, \lambda_k + 1, \dots, \lambda_{d+1}) \\ & \quad - 2\mu_{\lambda_1, \dots, \lambda_{d+1}} I_{\lambda_1, \dots, \lambda_{d+1}}(\lambda_1, \dots, \lambda_k + 1, \dots, \lambda_{d+1}) \\ & \quad - 2\nu_{\eta_1, \dots, \eta_{d+1}} I_{\eta_1, \dots, \eta_{d+1}}(\eta_1, \dots, \eta_{k-1}, \dots, \eta_{d+1}), \\ & \quad \text{sum over partition of } b \end{aligned} \tag{3-4}$$

where $\mu_{\lambda_1, \dots, \lambda_{d+1}}$ and $\nu_{\eta_1, \dots, \eta_{d+1}}$ are lagrange multipliers due to Kirchhoff's law on each subfractal, and also on each node, respectively. Minimizing the expression (3-4), we get linear equations between inner input flowing currents and lagrange multipliers together with the Kirchhoff's law for each subfractal and each vertex, respectively. Solving the equations thus obtained we can write all inner inward flowing currents as a linear function in terms of input ones. Substituting the expressions thus obtained for the inner currents in Eq. (3-4), we determine **TDP** of generation \mathbf{n} which is obviously a bilinear function of input currents with coefficients which are in general very involved functions of the connection resistances of the generation $\mathbf{n} - 1$. Comparing the final result with the expression (3-4), connection resistance of generation \mathbf{n} as its coefficient, we get the required transformation between connection resistances of generations \mathbf{n} and $\mathbf{n} - 1$, respectively:

$$R_{jk}(r+1) = f_{jk}(R_{lm}(r)_{m>l}), \quad k > j = 1, 2, \dots, d+1. \quad (3-5)$$

Here in this article, we show that the power and the anisotropy suppression exponents can be calculated, without having any knowledge of the explicit form of the functions $\mathbf{f}_{\mathbf{j}\mathbf{k}}$. All we need to know is some general properties of these functions which can be obtained rather easily from some physical requirements and also from dimensional analysis: these functions are homogeneous functions of degree one mapping positive connection resistances of generation $\mathbf{n} - \mathbf{1}$ into positive connection resistances of generation \mathbf{n} , that is they form positive homogeneous map of degree one.

All connection resistances are positive; none of them can be negative or zero. The physical reason behind it is that if, for example, the connection resistance $\mathbf{R}_{\mathbf{j}\mathbf{k}}$ becomes negative or if it vanishes, then for inward flowing currents $\mathbf{I}_{\mathbf{j}} = -\mathbf{I}_{\mathbf{k}} = \mathbf{I}$, and $\mathbf{I}_{\mathbf{l}} = \mathbf{0}$ if $\mathbf{l} \neq \mathbf{j} \neq \mathbf{k}$, we obviously get negative or zero power which is not physical in either cases. Analogously, we can rather easily deduce that none of them can be infinite, since all resistors of the network are finite, otherwise we will have infinite total dissipative power which is not again physical.

Definitely the transformation (3-5) is monotonically increasing, since by increasing the connection resistances at a given generation $\mathbf{n} - \mathbf{1}$, without changing the input currents, the total dissipative power of generation \mathbf{n} will increase, that is the connection resistances of generation \mathbf{n} will increase. Naturally, under the action of the point group $\mathbf{S}_{(\mathbf{d}+\mathbf{1})}$, [18] the connection resistances simply permute among themselves. For example, the exchange of the vertices \mathbf{j} and \mathbf{k} in $(\mathbf{d} + \mathbf{1})$ -simplex induces the following transformation among the connection resistances:

$$\begin{aligned} R_{jk} &\longrightarrow R_{kj} = R_{jk} \\ R_{jl} &\longrightarrow R_{lk} && \text{for } l \neq j \neq k \\ R_{kl} &\longrightarrow R_{lj} && \text{for } l \neq j \neq k \\ R_{lm} &\longrightarrow R_{lm} && \text{for } l \neq m \neq j \neq k. \end{aligned} \quad (3-6)$$

As an example, we give the explicit form of the transformation for the special case of $\mathbf{d} = \mathbf{2}$ and $\mathbf{b} = \mathbf{2}$

$$R_{12}(n) = \frac{R_{12}[R_{12}(n-1) + 2R_{13}(n-1) + 2R_{23}(n-1)]}{R_{12}(n-1) + R_{13}(n-1) + R_{23}(n-1)},$$

$$R_{13}(n) = \frac{R_{13}[R_{13}(n-1) + 2R_{12}(n-1) + 2R_{23}(n-1)]}{R_{12}(n-1) + R_{13}(n-1) + R_{23}(n-1)},$$

$$R_{23}(n) = \frac{R_{12}[R_{23}(n-1) + 2R_{12}(n-1) + 2R_{13}(n-1)]}{R_{12}(n-1) + R_{13}(n-1) + R_{23}(n-1)}.$$

4 Fixed Point of Recurrence Equation of Connection Resistances

In this section we present a rigorous proof that the renormalization group transformation of the connection resistances has a unique fixed direction in the space of connection resistances. That is, all of the flows stemming from the finite physical region of connection resistance space converge to infinity at a direction which has the same angle with all coordinates.

For simplicity we denote the connection resistances of generation $\mathbf{n} - \mathbf{1}$ $\mathbf{R}_{\mathbf{j}\mathbf{k}}(\mathbf{n} - \mathbf{1})$ ($\mathbf{k}, \mathbf{j} = \mathbf{1}, \mathbf{2}, \dots, \mathbf{d} + \mathbf{1}$), by \mathbf{X}_α ($\alpha = \mathbf{1}, \mathbf{2}, \dots, \frac{\mathbf{d}(\mathbf{d}+\mathbf{1})}{2}$) and the connection resistances of generation \mathbf{n} $\mathbf{R}_{\mathbf{j}\mathbf{k}}(\mathbf{n})$ ($\mathbf{k}, \mathbf{j} = \mathbf{1}, \mathbf{2}, \dots, \mathbf{d} + \mathbf{1}$), by \mathbf{X}'_α ($\alpha = \mathbf{1}, \mathbf{2}, \dots, \frac{\mathbf{d}(\mathbf{d}+\mathbf{1})}{2}$), respectively. Then the transformations (3-5) can be written as

$$X'_\alpha = f_\alpha(X_\beta) \quad \text{for } \alpha = 1, 2, \dots, \frac{d(d+1)}{2}. \quad (4-1)$$

Now we consider $\mathbf{X}_\alpha > \mathbf{0}$ ($\alpha = \mathbf{1}, \mathbf{2}, \dots, \frac{\mathbf{d}(\mathbf{d}+\mathbf{1})}{2}$) as coordinates of the interior points of a cone in $\frac{\mathbf{d}(\mathbf{d}+\mathbf{1})}{2}$ dimensional Euclidean space denoted by $\check{\mathcal{C}}$. Denoting the cone itself by \mathcal{C} , the transformation (4-1) can be considered as the map of this cone into itself:

$$\mathcal{C} \xrightarrow{F} \mathcal{C}, \quad (4-2)$$

where we have denoted the extension of the map (4-1) over the cone itself by \mathbf{F} .

From the action of the permutation group $\mathbf{S}_{(\mathbf{d}+\mathbf{1})}$ on the space of connection resistances (the cone \mathcal{C}) given in (3-6), it follows that the transformation (4-1) is equivariant with respect to the action of $\mathbf{S}_{(\mathbf{d}+\mathbf{1})}$, that is we have the following commutative diagram [25]:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{g} & \mathcal{C} \\ F \downarrow & & \downarrow F \\ \mathcal{C} & \xrightarrow{g} & \mathcal{C} \end{array} \quad \text{for every } g \in S_{(d+1)},$$

or

$$g(F(\mathcal{X})) = F(g(\mathcal{X})) \quad \text{for every } g \in S_{(d+1)} \text{ and } \mathcal{X} \in \mathcal{C}. \quad (4-3)$$

A cone in Euclidean space has the following properties[24]:

1. $\mathcal{C} + \mathcal{C} \subset \mathcal{C}$
2. $\lambda \mathcal{C} \subset \mathcal{C}$ for all $\lambda > \mathbf{0}$
3. $\mathcal{C} \cap -\mathcal{C} = \mathbf{0}$.

We denote interior of this cone by $\check{\mathcal{C}}$. One can define an "order relation" as follows:

$\mathcal{X} \geq \mathcal{Y}$ if $\mathcal{X} - \mathcal{Y} \in \mathcal{C}$ and $\mathcal{X} > \mathcal{Y}$ if $\mathcal{X} - \mathcal{Y} \in \check{\mathcal{C}}$. If one defines the numbers $\mathbf{m}(\mathcal{X}, \mathcal{Y})$ and $\mathbf{M}(\mathcal{X}, \mathcal{Y})$ such that

$$\begin{aligned} m(\mathcal{X}, \mathcal{Y}) &= \max_i \frac{x_i}{y_i} \\ M(\mathcal{X}, \mathcal{Y}) &= \min_i \frac{x_i}{y_i}, \end{aligned} \quad (4-4)$$

then for any $\mathcal{X}, \mathcal{Y} \in \check{\mathcal{C}}$ the following relation holds

$$m(\mathcal{X}, \mathcal{Y})\mathcal{Y} \leq \mathcal{X} \leq M(\mathcal{X}, \mathcal{Y})\mathcal{Y}. \quad (4-5)$$

The above relation and also all the other theorems of this section have been proved in reference [24]. Using the numbers defined in (4-4), one can define the Hilbert metric $\mathbf{d}(\mathcal{X}, \mathcal{Y})$ for any $\mathcal{X}, \mathcal{Y} \in \check{\mathcal{C}}$ as

$$d(\mathcal{X}, \mathcal{Y}) = \log\left[\frac{M(\mathcal{X}, \mathcal{Y})}{m(\mathcal{X}, \mathcal{Y})}\right] = \log\left[\max_{i,j} \frac{x_i y_j}{y_i x_j}\right], \quad (4-6)$$

with the usual property of a pseudometric on $\check{\mathcal{C}}$ and a metric on $\check{\mathcal{C}} \cap \mathbf{S}(\mathbf{0}, \mathbf{1})$, where $\mathbf{S}(\mathbf{0}, \mathbf{1})$ denotes the set of points of sphere of radius one with the origin as its center. The metric space $\check{\mathcal{C}} \cap \mathbf{S}(\mathbf{0}, \mathbf{1})$ is complete under Hilbert metric(4-6). Also, it is straightforward to see that the following assertions about this metric are valid :

1. For any $\mathcal{X}, \mathcal{Y} \in \check{\mathcal{C}}$ and $\mathbf{a}, \mathbf{b} \in \mathbf{R}$ we have

$$d(a\mathcal{X}, b\mathcal{X}) = d(\mathcal{X}, \mathcal{X}).$$

2. $\mathbf{d}(\mathcal{X}, \mathcal{X}) = 0$ if and only if $\mathcal{X} = \lambda \mathcal{X}$.
3. For any $\mathcal{X}, \mathcal{Y} \in \check{\mathcal{C}}$ the metric $\mathbf{d}(\mathcal{X}, \mathcal{Y})$ is finite. A given map of the cone into itself:

$$F : \mathcal{C} \longrightarrow \mathcal{C}$$

is called positive homogeneous of degree \mathbf{n} and monotonically increasing if, for all $\mathcal{X} \in \mathcal{C}$ and $\mathbf{a} > \mathbf{0}$, we have

$$F(a\mathcal{X}) = a^n F(\mathcal{X}),$$

and

$$\mathcal{X} \leq \mathcal{Y} \implies F(\mathcal{X}) \leq F(\mathcal{Y}).$$

According to the arguments given in section **III**, the transformation (3-5) or (4-1) is monotonically increasing, positive and homogeneous map of degree one.

Using the relation (4-5) one can prove that for a positive homogeneous map of degree one and monotonically increasing map like the transformation (4-1), the following inequality holds:

$$m(\mathcal{X}, \mathcal{Y})T(\mathcal{X}) \leq T(\mathcal{X}) \leq .M(\mathcal{X}), (\mathcal{Y})T(\mathcal{Y}). \quad (4-7)$$

It is straightforward to get the following inequality from the inequality (4-7)

$$m(\mathcal{X}, \mathcal{Y}) \leq m(T\mathcal{X}, T\mathcal{Y}) \leq M(\mathcal{X}, \mathcal{Y}) \leq M(T\mathcal{X}, T\mathcal{Y}). \quad (4-8)$$

From the above inequality and also from the definition of Hilbert metric (4-6), it follows that for all $\mathcal{X}, \mathcal{Y} \in \mathcal{C} \cap \mathbf{S}(\mathbf{0}, \mathbf{1})$ we have

$$d(T\mathcal{X}, T\mathcal{Y}) \leq d(\mathcal{X}, \mathcal{Y}). \quad (4-9)$$

Therefore, the transformation (4-1) satisfies the Lipschitz condition and is of contractive type. Thus, according to the Principle of Contraction Mapping Theorem, the contracting mapping (4-1) has a unique fixed point \mathcal{X}_0 in the complete metric space $(\mathcal{C} \cap \mathbf{S}(\mathbf{0}, \mathbf{1}), \mathbf{d})$ (\mathbf{d} is the Hilbert metric given in (4-4) and

$$\lim_{n \rightarrow \infty} \overbrace{F(F(\cdots F(F(\mathcal{X}))) \cdots)}^n = \mathcal{X}_0, \quad \text{for every } \mathcal{X} \in \mathcal{C}. \quad (4-10)$$

But, because of the equivariant property (4-3) of the transformation (4-1), any fixed point of the point group $\mathbf{S}_{(\mathbf{d}+\mathbf{1})}$ (or the stability point of the point group [25]) will definitely be the fixed point of the transformation (4-1) acting on the space $(\mathcal{C} \cap \mathbf{S}(\mathbf{0}, \mathbf{1}))$. Obviously, the direction $\mathbf{X}_1 = \mathbf{X}_2 = \cdots = \mathbf{X}_{\frac{\mathbf{d}(\mathbf{d}+\mathbf{1})}{2}}$ is the only fixed point of the permutation group $\mathbf{S}_{(\mathbf{d}+\mathbf{1})}$ acting on the space $(\mathcal{C} \cap \mathbf{S}(\mathbf{0}, \mathbf{1}))$. Hence, because of the uniqueness of the fixed point of the transformation (4-1) on the space $(\mathcal{C} \cap \mathbf{S}(\mathbf{0}, \mathbf{1}))$, the direction $\mathbf{X}_1 = \mathbf{X}_2 = \cdots = \mathbf{X}_{\frac{\mathbf{d}(\mathbf{d}+\mathbf{1})}{2}}$ is the only fixed direction of the connection resistances renormalization group transformation. This direction corresponds to the isotropic $(\mathbf{d} + \mathbf{1})$ -simplex, which indicates that the macroscopic conductivity becomes isotropic on large scales.

5 Determination of Inner Inward Flowing Currents of Subfractals in Isotropic State

In order to determine the inner inward flowing currents in terms of the input currents \mathbf{I}_j ($j = 1, 2, \dots, d+1$) in isotropic state, we have to minimize the **TDP** given in (3-4). But here in isotropic state all connection resistances are the same, hence they can be put equal to one in (3-4), simply by rescaling the lagrange multipliers of current conservations of vertices and subfractals. Now, by minimizing **TDP**, we get the following equation for \mathbf{I}

$$I_{\lambda_1, \dots, \lambda_{d+1}}(\lambda_1, \dots, \lambda_j + 1, \dots, \lambda_{d+1}) - \mu_{\lambda_1, \dots, \lambda_j, \lambda_{d+1}} - nu_{\lambda_1, \dots, \lambda_j + 1, \dots, \lambda_{j+1}} = 0 \quad (5-1)$$

together with the Kirchhoff's law for each subfractal and each vertex, respectively

$$\sum_{j=1}^{d+1} I_{\lambda_1, \dots, \lambda_j, \dots, \lambda_{d+1}}(\lambda_1, \dots, \lambda_j + 1, \dots, \lambda_{d+1}) = 0. \quad (5-2a)$$

$$\sum_{j=1}^{d+1} I_{\eta_1, \dots, \eta_j - 1, \dots, \eta_{d+1}}(\eta_1, \dots, \eta_j, \dots, \eta_{d+1}) = 0. \quad (5-2b)$$

We assume the following ansatz for the Lagrange multipliers:

$$\mu_{\lambda_1, \lambda_2, \dots, \lambda_{d+1}} = \sum_{k=1}^{d+1} a_{\lambda_1, \lambda_2, \dots, \lambda_{d+1}}(\lambda_k) I_k \quad (5-3a)$$

$$\nu_{\eta_1, \eta_2, \dots, \eta_{d+1}} = \sum_{k=1}^{d+1} b_{\eta_1, \eta_2, \dots, \eta_{d+1}}(\eta_k) I_k \quad (5-3b)$$

with $a_{\lambda_1, \lambda_2, \dots, \lambda_{d+1}}(0)$ and $b_{\eta_1, \eta_2, \dots, \eta_{d+1}}(0)$ taken to be zero.

Using the ansatz (5-3a) and (5-3b) in equation (5-1), the inflowing currents can be given in terms of \mathbf{a} and \mathbf{b} respectively, that is

$$I_{\lambda_1, \dots, \lambda_{d+1}}(\eta_1, \dots, \eta_{d+1}) = \sum_{k=1}^{d+1} a_{\lambda_1, \lambda_2, \dots, \lambda_{d+1}}(\lambda_k) I_k + \sum_{k=1}^{d+1} b_{\eta_1, \eta_2, \dots, \eta_{d+1}}(\eta_k) I_k. \quad (5-4)$$

Due to the $\mathbf{S}_{(d+1)}$ permutation symmetry of $(d+1)$ -simplex fractal, the parameters $\mathbf{a}_{\lambda_1, \lambda_2, \dots, \lambda_{d+1}}(\lambda_{\mathbf{k}})$ and $\mathbf{b}_{\eta_1, \eta_2, \dots, \eta_{d+1}}(\eta_{\mathbf{k}})$ depend only on the corresponding partition $\{\lambda_1, \lambda_2, \dots, \lambda_{d+1}\}$ and $\{\eta_1, \eta_2, \dots, \eta_{d+1}\}$, respectively. They do not change under the permutation of λ_i or η_i within a given partition. From now on, as far as a and b are concerned, only nonzero values are going to be quoted in their partition.

Actually one could write the currents in terms of input ones as in (5-4) by simply using the symmetry of simplex fractal, and the minimization of power is not required. Finally \mathbf{a} and \mathbf{b} can be determined through the equations (5-2a) and (5-2b). Obviously the number of equations are the same as the number of unknowns, hence the unknowns \mathbf{a} and \mathbf{b} can be determined uniquely. Here we determine the currents only for $\mathbf{b} = \mathbf{2}, \mathbf{3}, \mathbf{4}$ and $\mathbf{5}$, respectively.

Let us first consider the case where $\mathbf{b} = \mathbf{2}$

$$I_{0,0,\dots,0,\underbrace{1}_{j-th},0,\dots,0}(0,0,\dots,0,\underbrace{2}_{j-th},0,\dots,0) = I_j$$

$$I_{0,0,\dots,0,\underbrace{1}_{j-th},0,\dots,0}(0,0,\dots,0,\underbrace{1}_{j-th},0,\dots,0,\underbrace{1}_{k-th},0,\dots,0) = a_1(1)I_J + b_1(1)I_j + b(1)1I_k$$

Using equation (5-2b) we have

$$a_1(1) + 2b_1(1) = 0$$

and from equation (5-2a) we get

$$1 + da_1(1) + (d-1)b_1(1) = 0.$$

Solving the above equations we get the following result

$$I_{0,\dots,0,\underbrace{1}_{j-th},0,\dots,0}(0,\dots,0,\underbrace{1}_{j-th},0,\dots,0,\underbrace{1}_{k-th},0,\dots,0) = \frac{(I_k - I_j)}{(d+1)}.$$

Via the the procedure explained above, we can similarly calculate the inner inward flow-ing currents corresponding to $\mathbf{b} = \mathbf{3}, \mathbf{4}$ and $\mathbf{b} = \mathbf{5}$, where the details of calculation appear in Appendices I, II and III, respectively and below we quote only the results:

I: Inner inward flowing currents corresponding to decimation number $\mathbf{b} = \mathbf{3}$

$$I_{0,\dots,0,\underbrace{2}_{j-th},0,\dots,0}(0,\dots,0,\underbrace{2}_{j-th},0,\dots,0,\underbrace{1}_{k-th},0,\dots,0) =$$

$$-\frac{2d+5}{(2d+3)(d+1)}I_j + \frac{3}{(2d+3)(d+1)}I_k$$

$$I_{0,\dots,0,\underbrace{1}_{j-th},0,\dots,0,\underbrace{1}_{k-th},0,\dots,0}(0,\dots,0,\underbrace{2}_{j-th},0,\dots,0,\underbrace{1}_{k-th},0,\dots,0) =$$

$$\frac{2d+5}{(2d+3)(d+1)}I_j - \frac{3}{(2d+3)(d+1)}I_k$$

$$I_{0,\dots,0,\underbrace{1}_{j-th},0,\dots,0,\underbrace{1}_{k-th},0,\dots,0}(0,\dots,0,\underbrace{1}_{j-th},0,\dots,0,\underbrace{1}_{k-th},0,\dots,0) =$$

$$\frac{2}{(2d+3)(d+1)}(2I_l - I_j - I_k).$$

II: Inner inward flowing currents corresponding to decimation number $\mathbf{b} = 4$

$$I_{0,\dots,0,\underbrace{3}_{j-th},0,\dots,0}(0,\dots,0,\underbrace{3}_{j-th},0,\dots,0,\underbrace{1}_{k-th},0,\dots,0) =$$

$$-\frac{8d^3 + 52d^2 + 5(25d + 21)}{P}I_j + \frac{25d + 49}{P}I_k$$

$$I_{0,\dots,0,\underbrace{2}_{j-th},0,\dots,0,\underbrace{1}_{k-th},0,\dots,0}(0,\dots,0,\underbrace{3}_{j-th},0,\dots,0,\underbrace{1}_{k-th},0,\dots,0) =$$

$$\frac{8d^3 + 52d^2 + 125d + 105}{P}I_j - \frac{25d + 49}{P}I_k$$

$$I_{0,\dots,0,\underbrace{2}_{j-th},0,\dots,0,\underbrace{1}_{k-th},0,\dots,0}(0,\dots,0,\underbrace{2}_{j-th},0,\dots,0,\underbrace{2}_{k-th},0,\dots,0) =$$

$$-\frac{12d^2 + 79d + 91}{P}(I_j - I_k)$$

$$I_{0,\dots,0,\underbrace{2}_{j-th},0,\dots,0,\underbrace{1}_{k-th},0,\dots,0}(0,\dots,0,\underbrace{2}_{j-th},0,\dots,0,\underbrace{1}_{k-th},0,\dots,0,\underbrace{1}_{l-th},0,\dots,0) =$$

$$-8\frac{d^2 + 6d + 7}{P}I_j - 4\frac{3d + 7}{P}I_k + 2\frac{19d + 35}{P}I_l$$

$$I_{0,\dots,0,\underbrace{1}_{j-th},0,\dots,0,\underbrace{1}_{k-th},0,\dots,0,\underbrace{1}_{l-th},0,\dots,0}(0,\dots,0,\underbrace{2}_{j-th},0,\dots,0,\underbrace{1}_{k-th},0,\dots,0,\underbrace{1}_{l-th},0,\dots,0) =$$

$$16\frac{d^2 + 6d + 7}{P}I_j - 2\frac{13d + 21}{P}(I_k + I_l)$$

$$I_{0,\dots,0,\underbrace{1}_{j-th},0,\dots,0,\underbrace{1}_{k-th},0,\dots,0,\underbrace{1}_{l-th},0,\dots,0}(0,\dots,0,\underbrace{1}_{j-th},0,\dots,0,\underbrace{1}_{k-th},0,\dots,0,\underbrace{1}_{l-th},0,\dots,0,\underbrace{1}_{m-th},0,\dots,0) =$$

$$-4\frac{4d+7}{P}(I_j + I_k + I_l - 3I_m)$$

where P is defined as

$$P = (8d^3 + 44d^2 + 81d + 49)(d + 1).$$

III: Inner inward flowing currents corresponding to decimation number $\mathbf{b} = \mathbf{5}$

$$\begin{aligned}
& I_{0,\dots,0,\underbrace{4}_{j-th},0,\dots,0(0,\dots,0,\underbrace{4}_{j-th},0,\dots,0,\underbrace{1}_{k-th},0,\dots,0) = \\
& \quad \frac{192d^6 + 2720d^5 + 16332d^4 + 53648d^3 + 102215d^2 + 106746d + 47255}{Q}I_j \\
& \quad - 3\frac{542d^3 + 3803d^2 + 8576d + 6275}{Q}I_k \\
& I_{0,\dots,0,\underbrace{3}_{j-th},0,\dots,0,\underbrace{1}_{k-th},0,\dots,0(0,\dots,0,\underbrace{4}_{j-th},0,\dots,0,\underbrace{1}_{k-th},0,\dots,0) = \\
& \quad - \frac{192d^6 + 2720d^5 + 16332d^4 + 53648d^3 + 102215d^2 + 106746d + 47255}{Q} \\
& \quad + 3\frac{542d^3 + 3803d^2 + 8576d + 6275}{Q}I_k \\
& I_{0,\dots,0,\underbrace{3}_{j-th},0,\dots,0,\underbrace{1}_{k-th},0,\dots,0(0,\dots,0,\underbrace{3}_{j-th},0,\dots,0,\underbrace{2}_{k-th},0,\dots,0) = \\
& \quad \frac{288d^5 + 4104d^4 + 24466d^3 + 70139d^2 + 94822d + 48213}{Q}I_j \\
& \quad - \frac{600d^4 + 8602d^3 + 36869d^2 + 62290d + 36303}{Q}I_k \\
& I_{0,\dots,0,\underbrace{3}_{j-th},0,\dots,0,\underbrace{1}_{k-th},0,\dots,0(0,\dots,0,\underbrace{3}_{j-th},0,\dots,0,\underbrace{1}_{k-th},0,\dots,0,\underbrace{1}_{l-th},0,\dots,0) = \\
& \quad 2\frac{96d^5 + 1312d^4 + 7426d^3 + 20691d^2 + 27782d + 14261}{Q}I_j \\
& \quad + 2\frac{300d^3 + 2462d^2 + 6245d + 5043}{Q}I_k \\
& \quad - 2\frac{1326d^3 + 8947d^2 + 19483d + 13782}{Q}I_l
\end{aligned}$$

$$\begin{aligned}
& I_{0,\dots,0,\underbrace{2}_{j-th},0,\dots,0,\underbrace{2}_{k-th},0,\dots,0}(0,\dots,0,\underbrace{3}_{j-th},0,\dots,0,\underbrace{2}_{k-th},0,\dots,0) = \\
& \quad - \frac{288d^5 + 4104d^4 + 24466d^3 + 70139d^2 + 94822d + 48213}{Q} I_j \\
& \quad - \frac{600d^4 + 8602d^3 + 36869d^2 + 62290d + 36303}{Q} I_k \\
& I_{0,\dots,0,\underbrace{2}_{j-th},0,\dots,0,\underbrace{2}_{k-th},0,\dots,0}(0,\dots,0,\underbrace{2}_{j-th},0,\dots,0,\underbrace{2}_{k-th},0,\dots,0,\underbrace{1}_{l-th},0,\dots,0) = \\
& \quad 2 \frac{144d^4 + 1896d^3 + 8260d^2 + 14607d + 9059}{Q} I_j \\
& \quad + \frac{144d^4 + 1896d^3 + 8260d^2 + 14607d + 9059}{Q} I_k \\
& \quad - 2 \frac{784d^3 + 5144d^2 + 10907d + 7507}{Q} I_l \\
& I_{0,\dots,0,\underbrace{2}_{j-th},0,\dots,0,\underbrace{1}_{k-th},0,\dots,0,\underbrace{1}_{l-th},0,\dots,0}(0,\cdot 0,\underbrace{3}_{j-th},0,\cdot 0,\underbrace{1}_{k-th},0,\cdot 0,\underbrace{1}_{l-th},0,\cdot 0) = \\
& \quad 2 \frac{1026d^3 + 6485d^2 + 13238d + 8739}{Q} (I_k + I_l) \\
& \quad - 4 \frac{96d^5 + 1312d^4 + 7426d^3 + 20691d^2 + 27782d + 14261}{Q} I_j \\
& I_{0,\dots,0,\underbrace{2}_{j-th},0,\dots,0,\underbrace{1}_{k-th},0,\dots,0,\underbrace{1}_{l-th},0,\dots,0}(0,\cdot 0,\underbrace{2}_{j-th},0,\cdot 0,\underbrace{2}_{k-th},0,\cdot 0,\underbrace{1}_{l-th},0,\cdot 0) = \\
& \quad 2 \frac{312d^4 + 4084d^3 + 16326d^2 + 26083d + 14489}{Q} I_j \\
& \quad - 2 \frac{228d^4 + 2990d^3 + 12293d^2 + 20345d + 11774}{Q} I_k \\
& \quad + \frac{784d^3 + 5144d^2 + 10907d + 7507}{Q} I_l \\
& I_{0,\dots,0,\underbrace{2}_{j-th},0,\dots,0,\underbrace{1}_{k-th},0,\dots,0,\underbrace{1}_{l-th},0,\dots,0}(0,\dots,0,\underbrace{2}_{j-th},0,\dots,0,\underbrace{1}_{k-th},0,\dots,0,\underbrace{1}_{l-th},0,\cdot,\underbrace{1}_{m-th},0,\dots,0) = \\
& \quad 8 \frac{48d^4 + 596d^3 + 2389d^2 + 3899d + 2238}{Q} I_j \\
& \quad + 6 \frac{152d^3 + 1062d^2 + 2361d + 1691}{Q} I_k \\
& \quad + \frac{152d^3 + 1062d^2 + 2361d + 1691}{Q} I_l
\end{aligned}$$

$$\begin{aligned}
& -2 \frac{316d^3 + 2041d^2 + 4273d + 2908}{Q} I_m \\
& I_{0,\dots,0,\underbrace{1}_{j-th},0,\dots,0,\underbrace{1}_{k-th},0,\dots,0,\underbrace{1}_{l-th},0,\dots,0,\underbrace{1}_{m-th},0,\dots,0,\underbrace{2}_{j-th},0,\dots,0,\underbrace{1}_{k-th},0,\dots,0,\underbrace{1}_{l-th},0,\dots,0,\underbrace{1}_{m-th},0,\dots,0) = \\
& -24 \frac{48d^4 + 596d^3 + 2389d^2 + 3899d + 2238}{Q} I_j \\
& - \frac{164d^3 + 979d^2 + 1912d + 1217}{Q} (I_k + I_l + I_m) \\
& I_{0,\dots,0,\underbrace{1}_{j-th},0,\dots,0,\underbrace{1}_{k-th},0,\dots,0,\underbrace{1}_{l-th},0,\dots,0,\underbrace{1}_{m-th},0,\dots,0,\underbrace{1}_{j-th},0,\dots,0,\underbrace{1}_{k-th},0,\dots,0,\underbrace{1}_{l-th},0,\dots,0,\underbrace{1}_{m-th},0,\dots,0,\underbrace{1}_{n-th},0,\dots,0) = \\
& 12 \frac{96d^3 + 604d^2 + 1237d + 825}{Q} (I_j + I_k + I_l + I_m - 4I_n)
\end{aligned}$$

where Q is defined as

$$Q = 192d^7 + 2720d^6 + 16332d^5 + 53648d^4 + 103841d^3 + 118155d^2 + 72983d + 18825.$$

6 Scaling Exponent of Anisotropy Suppression of $(d+1)$ -Simplex Fractal Conductor Network

To investigate the abolition of anisotropy and also in order to calculate the scaling exponent of its suppression, we linearize the recursion map (4-1) near the fixed direction (isotropy state) of this map:

$$\lim_{n \rightarrow \infty} R_{jk}(n) = R \quad \text{for } k > j = 1, 2, \dots, d+1.$$

This leads us to write

$$R_{jk}(n) = R + \varepsilon_{jk}(n); \quad \text{for } k > j = 1, 2, \dots, d+1 \quad (6-1)$$

with $\varepsilon_{jk}(\mathbf{n})$ as an infinitesimal deviation of the connection resistances of generation \mathbf{n} from the isotropic state, for large values of \mathbf{n} . Now, all we need to know is the recursion relations between the deviation of connection resistances of the generation \mathbf{n} and the infinitesimal deviation of connection resistances of the generation $\mathbf{n} - \mathbf{1}$, for large values of \mathbf{n} .

These recursion relations can easily be obtained, if we compare the deviation of **TDP** from the isotropic state of generation \mathbf{n} with the deviation of the same quantity, calculated as the sum of deviation of **TDP** of its subfractals of generation $\mathbf{n} - \mathbf{1}$. Clearly the deviation of **TDP**

of generation \mathbf{n} can be obtained from the expression (3-3), provided that in (3-3) we replace the connection resistances with the deviation of the connection resistances of generation \mathbf{n} . Also **TDP** of generation \mathbf{n} is the sum of **TDP** of subfractals generations $\mathbf{n} - \mathbf{1}$, where again the latter can be obtained from (3-3), if we replace in (3-3) the input currents with the inner inflowing currents (which have been expressed in terms of input currents in section V) and the connection resistances with the deviation of the generation $\mathbf{n} - \mathbf{1}$, respectively. Proceeding as above we obtain the recursion relations of the following form for the deviation of connection resistances of generations $\mathbf{n} - \mathbf{1}$ and \mathbf{n} in a $(\mathbf{d} + \mathbf{1})$ -simplex fractal conductor network, for large values of \mathbf{n} :

$$_{j \neq k} \varepsilon_{jk}(n) = f(d, b)_{j \neq k} \varepsilon_{jk}(n - 1) + g(d, b) \left(\sum_{l=1 \neq j \neq k}^{d+1} \varepsilon_{jl}(n - 1) + \sum_{l=1 \neq j \neq k}^{d+1} \varepsilon_{lk}(n - 1) \right) + g(d, b) \left(\sum_{l \neq m=1 \neq j \neq k}^{d+1} \varepsilon_{lm}(n - 1) \right). \quad (6-2)$$

For a given value of $\mathbf{j} \neq \mathbf{k}$ we denote $\varepsilon_{\mathbf{jk}}(\mathbf{n})(\varepsilon_{jk}(n - 1))$ by $X(X')$. Next we assume that $\varepsilon_{\mathbf{jl}}(\mathbf{n})(\varepsilon_{jl}(\mathbf{n} - \mathbf{1}))$ and $\varepsilon_{\mathbf{lk}}(\mathbf{n})(\varepsilon_{lk}(\mathbf{n} - \mathbf{1}))$ with $(\mathbf{1} \neq \mathbf{j} \neq \mathbf{k})$ are all equal which are denoted by $\mathbf{Y}(\mathbf{Y}')$. Finally we assume that the remaining deviation of connection resistances, that is, $\varepsilon_{\mathbf{lm}}(\mathbf{n})(\varepsilon_{lm}(\mathbf{n} - \mathbf{1}))$ with $(\mathbf{m} \neq \mathbf{l} \neq \mathbf{j} \neq \mathbf{k})$ are all equal which are denoted by $\mathbf{Z}(\mathbf{Z}')$. Then the recursion relations (6-2) take the following form:

$$\begin{pmatrix} f(d, b) & 2(d - 1)g(d, b) & (d - 2)(d - 1)h(d, b) \\ g(d, b) & f + g(d - 1) + 2h(d - 2) & g(d - 2) + h(d - 3)(d - 2) \\ 2h(d, b) & 4g(d, b) + 4(d - 3)h(d, b) & f(d, b) + 2(d - 3)g(d, b) + (d - 4)(d - 3)h(d, b) \end{pmatrix} \begin{pmatrix} X' \\ Y' \\ Z' \end{pmatrix} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}. \quad (6 - 3)$$

The following eigen-values are obtained by diagonalizing the $\mathbf{3} \times \mathbf{3}$ matrix (6-3) . The eigen-values are quoted in decreasing order as:

$$\begin{aligned} \lambda_{max} &= \frac{d^2 h(d, b) + d(4g(d, b) - 3h(d, b)) + 2f(d, b) - 4g(d, b) + 2h(d, b)}{2} \\ \lambda_{middle} &= d(g(d, b) - h(d, b)) + f(d, b) - 3g(d, b) + 2h(d, b) \\ \lambda_{min} &= f(d, b) - 2g(d, b) + h(d, b), \end{aligned} \quad (6-3)$$

where the corresponding eigen-vectors are given as the rows of the following matrix:

$$\begin{pmatrix} 1 & 1 & 1 \\ 2(d-1) & d-3 & -2 \\ (d-1)(d-2) & -(d-2) & 2 \end{pmatrix}. \quad (6-4)$$

We see that the eigen-directions do not depend on the decimation number \mathbf{b} but rather only on the dimension \mathbf{d} . This is again due to equivariancy of the map (4-1) with respect to the action of the point group $\mathbf{S}_{(\mathbf{d}+1)}$ on the space of the connection resistances given in (4-3).

As expected, the maximum eigen-value corresponds to isotropy state, which gives the power scaling exponent of $(\mathbf{d} + 1)$ -simplex fractal conductor networks [14, 15, 16, 26]. Therefore, anisotropy vanishes with a scaling exponent which can be obtained in terms of the eigenvalues in the following way: from the recurrence relation (3-5) and its linearized form (6-2), it follows that, for large values of \mathbf{n} we have

$$\lim_{n \rightarrow \infty} R_{jk} \sim L_n^{D_2}, \quad \text{for every } j \neq k = 1, 2, \dots, d+1 \quad (6-5)$$

where $\mathbf{L}_n = \mathbf{b}^n$, and the power scaling exponent \mathbf{D}_2 is defined as:

$$D_2(d, b) = \frac{\log \lambda_{max}}{\log b} \quad (6-6)$$

and

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{R_{jk}}{R_{jl}} - 1 &\sim L_n^{\bar{\lambda}} \quad \text{for every } j \neq k \neq l = 1, 2, \dots, d+1 \\ \lim_{n \rightarrow \infty} \frac{R_{jk}}{R_{lm}} - 1 &\sim L_n^{\bar{\lambda}} \quad \text{for every } j \neq k \neq l \neq m = 1, 2, \dots, d+1 \end{aligned} \quad (6-7)$$

where the scaling exponent of suppression of the anisotropy, $\bar{\lambda}$, is defined as

$$\bar{\lambda}(d, b) = \frac{\log \frac{\text{maximum eigenvalue}}{\text{the next greatest eigenvalue}}}{\log b} = \frac{\log \frac{\lambda_{max}}{\lambda_{middle}}}{\log b}. \quad (6-8)$$

In the remaining part of this section we quote the results for $\mathbf{b} = \mathbf{2}, \mathbf{3}, \mathbf{4}$ and $\mathbf{5}$, respectively.

I : $\mathbf{b} = \mathbf{2}$

$$\begin{aligned} f(d, 2) &= \frac{5d+3}{(d+1)^2} \\ g(d, 2) &= \frac{d-1}{(d+1)^2} \end{aligned}$$

$$h(d, 2) = \frac{-2}{(d+1)^2}$$

$$D_2(d, 2) = \frac{\log \frac{d+3}{d+1}}{\log 2}$$

$$\bar{\lambda}(d, 2) = \frac{\log \frac{d+3}{d+2}}{\log 2}$$

(6-9)

II : b = 3

$$f(d, 3) = \frac{(8d^3 + 88d^2 + 145d + 59)}{(2d+3)^2(d+1)^2}$$

$$g(d, 3) = \frac{4d^3 + 20d^2 + d - 24}{(2d+3)^2(d+1)^2}$$

$$h(d, 3) = \frac{-(4d^2 + 24d + 25)}{(2d+3)^2(d+1)^2}$$

$$D_2(d, 3) = \frac{\log \frac{2d^2+9d+19}{(2d+3)(d+1)}}{\log 3}$$

$$\bar{\lambda}(d, 3) = \frac{\log \frac{(2d+3)(2d^2+9d+19)}{4d^3+20d^2+41d+31}}{\log 3}$$

(6-10)

III:b=4

$$f(d, 4) = \frac{128d^7 + 1792d^6 + 13936d^5 + 59116d^4 + 137757d^3 + 175421d^2 + 113267d + 28567}{(8d^4 + 52d^3 + 125d^2 + 130d + 49)^2}$$

$$g(d, 4) = \frac{64d^7 + 832d^6 + 4768d^5 + 13448d^4 + 16313d^3 - 701d^2 - 18501d - 11319}{(8d^4 + 52d^3 + 125d^2 + 130d + 49)^2}$$

$$h(d, 4) = \frac{-(64d^6 + 896d^5 + 5664d^4 + 19096d^3 + 35061d^2 + 32970d + 12397)}{(8d^4 + 52d^3 + 125d^2 + 130d + 49)^2}$$

$$D_2(d, 4) = \frac{\log \frac{8d^4+68d^3+253d^2+588d+539}{8d^4+52d^3+125d^2+130d+49}}{\log 4}$$

$$\bar{\lambda}(d, 4) = \frac{\log \frac{8d^4+68d^3+253d^2+588d+539}{8d^4+52d^3+125d^2+130d+49}}{\log 4}$$

(6-11)

IV:b=5

$$f(d, 5) = \frac{1}{Q^2}(73728d^{13} + 2162688d^{12} + 29576192d^{11} + 258155264d^{10} + 1614743456d^9 \\ + 7530179904d^8 + 26333589428d^7 + 68567523880d^6 + 131200269465d^5 \\ + 180815964435d^4 + 173716650934d^3 + 109891587638d^2 + 40940417277d \\ + 6768087791)$$

$$g(d, 5) = \frac{1}{Q^2}(36864d^{13} + 1044480d^{12} + 13706752d^{11} + 110574336d^{10} + 607189008d^9 \\ + 2357625920d^8 + 6492213656d^7 + 12314821608d^6 + 14686625629d^5 \\ + 7431447086d^4 - 6360742466d^3 - 13852397352d^2 - 9571181547d \\ - 2499415382)$$

$$h(d, 5) = \frac{-1}{Q^2}(36864d^{12} + 1081344d^{11} + 14788096d^{10} + 125353216d^9 + 732200464d^8 \\ + 3083843024d^7 + 9521524696d^6 + 21544010108d^5 + 35237710633d^4 \\ + 40459001364d^3 + 30870831766d^2 + 14032614408d + 2871299265)$$

$$D_2(d, 5) = \frac{\log \frac{192d^7 + 3104d^6 + 22348d^5 + 95720d^4 + 280525d^3 + 559419d^2 + 652155d + 320017}{Q}}{\log 5}$$

$$\bar{\lambda}(d, 5) = \frac{\log \frac{(192d^7 + 3104d^6 + 22348d^5 + 95720d^4 + 280525d^3 + 559419d^2 + 652155d + 320017)Q}{(d+1)P}}{\log 5}$$

(6-12)

with **P** and **Q** defined as

$$P = (36864d^{13} + 1044480d^{12} + 13706752d^{11} + 110574336d^{10} + 614600976d^9 \\ + 2499189440d^8 + 7689210552d^7 + 18190236812d^6 + 33121369305d^5 \\ + 45749851193d^4 + 46378189714d^3 + 32473949342d^2 + 13985550557d \\ + 2781136877)$$

and

$$Q = (192d^6 + 2528d^5 + 13804d^4 + 39844d^3 + 63997d^2 + 54158d + 18825)(d + 1).$$

It is straightforward to see that these results will also hold true for $(\mathbf{d} + \mathbf{1})$ -honeycomb fractal conductor network with decimation number \mathbf{b} , which can be constructed from a given $(\mathbf{d} + \mathbf{1})$ -simplex fractal conductor network, simply by replacing the resistors in the links with the resistors which connect the center of a subfractal to its vertices, (see Fig. 2) where, this has also been shown for $\mathbf{d} = \mathbf{2}$ and $\mathbf{b} = \mathbf{2}$ case in reference [16].

Appendix I: Calculation of currents of $\mathbf{b} = \mathbf{3}$.

Here in this Appendix we give the detail of calculation of inner inward flowing currents corresponding to decimation number $\mathbf{b} = \mathbf{3}$

Following the procedure of section IV, for $\mathbf{b} = \mathbf{3}$ we have

$$\begin{aligned}
I_{0,\dots,0,\underbrace{2}_{j-th},0,\dots,0(0,\dots,0,\underbrace{3}_{j-th},0,\dots,0) &= I_j \\
I_{0,\dots,0,\underbrace{2}_{j-th},0,\dots,0(0,\dots,0,\underbrace{2}_{j-th},0,\dots,0,\underbrace{1}_{k-th},0,\dots,0) &= \\
& a_2(2)I_j + b_{21}(2)I_j + b_{21}(1)I_k \\
I_{0,\dots,0,\underbrace{1}_{j-th},0,\dots,0,\underbrace{1}_{k-th},0,\dots,0(0,\dots,0,\underbrace{2}_{j-th},0,\dots,0,\underbrace{1}_{k-th},0,\dots,0) &= \\
& a_{11}(1)(I_j + I_k) + b_{21}(2)I_j + b_{21}(1)I_k \\
I_{0,\dots,0,\underbrace{1}_{j-th},0,\dots,0,\underbrace{1}_{k-th},0,\dots,0(0,\dots,0,\underbrace{1}_{j-th},0,\dots,0,\underbrace{1}_{k-th},0,\dots,0,\underbrace{1}_{l-th},0,\dots,0) &= \\
& a_{11}(1)(I_j + I_k) + b_{111}(1)(I_j + I_k + I_l).
\end{aligned}$$

Using equation (5-2a) in subfractal $(\mathbf{0}, \dots, \mathbf{0}, \underbrace{\mathbf{2}}_{j-th}, \mathbf{0}, \dots, \mathbf{0})$, we get

$$1 + d(a_2(2) + b_{21}(2)) - b_{21}(1) = 0,$$

also using equation (5-2a) in subfractal $(\mathbf{0}, \dots, \mathbf{0}, \mathbf{1}_j, \mathbf{0}, \dots, \mathbf{1}_k, \mathbf{0}, \dots, \mathbf{0})$ we get

$$(d + 1)a_{11}(1) + b_{21}(1) + b_{21}(2) + (d - 2)b_{111}(1) = 0,$$

also, for vertices equation (5-2b) gives

$$a_2(2) + 2b_{21}(2) + a_{11}(1) = 0$$

$$a_{11}(1) + 2b_{21}(1) = 0$$

$$2a_{11}(1) + 3b_{111}(1) = 0.$$

By solving the above equations we can determine inner inward flowing currents corresponding to decimation number $\mathbf{b} = \mathbf{3}$ which is given in section V.

Appendix II: Calculation of currents of $\mathbf{b} = \mathbf{4}$.

Here in this Appendix we give the detail of calculation of inner inward flowing currents corresponding to decimation number $\mathbf{b} = \mathbf{4}$

Similarly, following the procedure of section IV, for $\mathbf{b} = \mathbf{4}$ we have

$$\begin{aligned}
I_{0,\dots,0,\underbrace{3}_{j-th},0,\dots,0(0,\dots,0,\underbrace{4}_{j-th},0,\dots,0) &= I_j \\
I_{0,\dots,0,\underbrace{3}_{j-th},0,\dots,0(0,\dots,0,\underbrace{3}_{j-th},0,\dots,0,\underbrace{1}_{k-th},0,\dots,0) &= \\
a_3(3)I_j + b_{31}(3)I_j + b_{31}(1)I_k \\
I_{0,\dots,0,\underbrace{2}_{j-th},0,\dots,0,\underbrace{1}_{k-th},0,\dots,0(0,\dots,0,\underbrace{3}_{j-th},0,\dots,0,\underbrace{1}_{k-th},0,\dots,0) &= \\
a_{21}(2)I_j + a_{21}(1)I_k + b_{31}(3)I_j + b_{31}(1)I_k \\
I_{0,\dots,0,\underbrace{2}_{j-th},0,\dots,0,\underbrace{1}_{k-th},0,\dots,0(0,\dots,0,\underbrace{2}_{j-th},0,\dots,0,\underbrace{2}_{k-th},0,\dots,0) &= \\
a_{21}(2)I_j + a_{21}(1)I_k + b_{22}(2)(I_j + I_k) \\
I_{0,\dots,0,\underbrace{2}_{j-th},0,\dots,0,\underbrace{1}_{k-th},0,\dots,0(0,\dots,0,\underbrace{2}_{j-th},0,\dots,0,\underbrace{1}_{k-th},0,\dots,0,\underbrace{1}_{l-th},0,\dots,0) &= \\
a_{21}(2)I_j + a_{21}(1) + b_{211}(2)I_j + b_{211}(1)(I_k + I_l) \\
I_{0,\dots,0,\underbrace{1}_{j-th},0,\dots,0,\underbrace{1}_{k-th},0,\dots,0,\underbrace{1}_{l-th},0,\dots,0(0,\dots,0,\underbrace{2}_{j-th},0,\dots,0,\underbrace{1}_{k-th},0,\dots,0,\underbrace{1}_{l-th},0,\dots,0) &= \\
a_{111}(1)(I_j + I_k + I_l) + b_{211}(2)I_j + b_{211}(1)(I_k + I_l)
\end{aligned}$$

$$I_{0,\dots,0,\underbrace{1}_{j-th},0,\dots,0,\underbrace{1}_{k-th},0,\dots,0,\underbrace{1}_{l-th},0,\dots,0(0,\dots,0,\underbrace{1}_{j-th},0,\dots,0,\underbrace{1}_{k-th},0,\dots,0,\underbrace{1}_{l-th},0,\dots,0,\underbrace{1}_{m-th},0,\dots,0) =$$

$$a_{111}(1)(I_j + I_k + I_l) + b_{1111}(1)(I_j + I_k + I_l + I_m).$$

Now, imposing Kirchhoff's law on subfractals and vertices, we get the following equations for **a** and **b**

$$\begin{aligned} 1 + da_3(3) + b_{31}(3) - b_{31}(1) &= 0 \\ (d+1)a_{21}(2) + b_{31}(3) + b_{22}(2) + (d-1)b_{211}(2) - b_{211}(1) &= 0 \\ (d+1)a_{21}(2) + b_{31}(1) + b_{22}(2) + (d-2)b_{211}(1) &= 0 \\ (d+1)a_{111}(2) + b_{211}(2) + (d-3)b_{1111}(1) &= 0 \\ a_{21}(1) + 2b_{31}(1) &= 0 \\ a_3(3) + b_{21}(2) + 2b_{31}(3) &= 0 \\ a_{21}(2) + a_{21}(1) + 2b_{22}(2) &= 0 \\ 2a_{21}(2) + a_{111}(1) + 3b_{211}(2) &= 0 \\ a_{21}(1) + a_{111}(1) + 3b_{211}(1) &= 0 \\ 3a_{111}(1) + 4b_{1111}(1) &= 0. \end{aligned}$$

By solving the above equations we can determine inner inward flowing currents corresponding to decimation number **b = 4** which appear in section V.

Appendix III: Calculation of currents of **b = 5**.

Here in this Appendix we give the detail of calculation of inner inward flowing currents corresponding to decimation number **b = 5**

Finally following the procedure of section IV, for **b = 5** we have

$$I_{0,\dots,0,\underbrace{4}_{j-th},0,\dots,0(0,\dots,0,\underbrace{5}_{j-th},0,\dots,0) = I_j$$

$$I_{0,\dots,0,\underbrace{4}_{j-th},0,\dots,0(0,\dots,0,\underbrace{4}_{j-th},0,\dots,0,\underbrace{1}_{k-th},0,\dots,0) =$$

$$a_4(4)I_j + b_{41}(4)I_j + b_{41}(1)I_k$$

$$I_{0,\dots,0,\underbrace{3}_{j-th},0,\dots,0,\underbrace{1}_{k-th},0,\dots,0}(0,\dots,0,\underbrace{4}_{j-th},0,\dots,0,\underbrace{1}_{k-th},0,\dots,0) =$$

$$a_{31}(3)I_j + a_{31}(1)I_k + b_{41}(4)I_j + b_{41}(1)I_k$$

$$I_{0,\dots,0,\underbrace{3}_{j-th},0,\dots,0,\underbrace{1}_{k-th},0,\dots,0}(0,\dots,0,\underbrace{3}_{j-th},0,\dots,0,\underbrace{2}_{k-th},0,\dots,0) =$$

$$a_{31}(3)I_j + a_{31}(1)I_k + b_{32}(3)I_j + b_{32}(3)I_k$$

$$I_{0,\dots,0,\underbrace{3}_{j-th},0,\dots,0,\underbrace{1}_{k-th},0,\dots,0}(0,\dots,0,\underbrace{3}_{j-th},0,\dots,0,\underbrace{1}_{k-th},0,\dots,0,\underbrace{1}_{l-th},0,\dots,0) =$$

$$a_{31}(3)I_j + a_{31}(1) + b_{311}(3)I_j + b_{311}(1)(I_k + I_l)$$

$$I_{0,\dots,0,\underbrace{2}_{j-th},0,\dots,0,\underbrace{2}_{k-th},0,\dots,0}(0,\dots,0,\underbrace{3}_{j-th},0,\dots,0,\underbrace{2}_{k-th},0,\dots,0) =$$

$$a_{22}(2)(I_j + I_k) + b_{32}(3)I_j + b_{32}(2)I_k$$

$$I_{0,\dots,0,\underbrace{2}_{j-th},0,\dots,0,\underbrace{2}_{k-th},0,\dots,0}(0,\dots,0,\underbrace{2}_{j-th},0,\dots,0,\underbrace{2}_{k-th},0,\dots,0,\underbrace{1}_{l-th},0,\dots,0) =$$

$$a_{22}(2)(I_j + I_k) + b_{221}(2)(I_j + I_k) + b_{221}(1)I_l$$

$$I_{0,\dots,0,\underbrace{2}_{j-th},0,\dots,0,\underbrace{1}_{k-th},0,\dots,0,\underbrace{1}_{l-th},0,\dots,0}(0,\dots,0,\underbrace{3}_{j-th},0,\dots,0,\underbrace{1}_{k-th},0,\dots,0,\underbrace{1}_{l-th},0,\dots,0) =$$

$$a_{211}(2)I_j + a_{211}(1)(I_k + I_l) + b_{311}(3)I_j + b_{311}(1)(I_k + I_l)$$

$$I_{0,\dots,0,\underbrace{2}_{j-th},0,\dots,0,\underbrace{1}_{k-th},0,\dots,0,\underbrace{1}_{l-th},0,\dots,0}(0,\dots,0,\underbrace{2}_{j-th},0,\dots,0,\underbrace{2}_{k-th},0,\dots,0,\underbrace{1}_{l-th},0,\dots,0) =$$

$$a_{211}(2)I_j + a_{211}(1)(I_k + I_l) + b_{221}(2)(I_j + I_k) + b_{221}(1)I_l$$

$$I_{0,\dots,0,\underbrace{2}_{j-th},0,\dots,0,\underbrace{1}_{k-th},0,\dots,0,\underbrace{1}_{l-th},0,\dots,0}(0,\dots,0,\underbrace{2}_{j-th},0,\dots,0,\underbrace{1}_{k-th},0,\dots,0,\underbrace{1}_{l-th},0,\dots,0,\underbrace{1}_{m-th},0,\dots,0) =$$

$$a_{211}(2)I_j + a_{211}(1)(I_k + I_l) + b_{2111}(2)I_j + b_{2111}(1)(I_k + I_l + I_m)$$

$$I_{0,\dots,0,\underbrace{1}_{j-th},0,\dots,0,\underbrace{1}_{k-th},0,\dots,0,\underbrace{1}_{l-th},0,\dots,0,\underbrace{1}_{m-th},0,\dots,0}(0,\dots,0,\underbrace{2}_{j-th},0,\dots,0,\underbrace{1}_{k-th},0,\dots,0,\underbrace{1}_{l-th},0,\dots,0,\underbrace{1}_{m-th},0,\dots,0) =$$

$$a_{111}(1)(I_j + I_k + I_l + I_m) + b_{2111}(2)I_j + b_{2111}(1)(I_k + I_l + I_m)$$

$$I_{0,\dots,0,\underbrace{1}_{j-th},0,\dots,0,\underbrace{1}_{k-th},0,\dots,0,\underbrace{1}_{l-th},0,\dots,0,\underbrace{1}_{m-th},0,\dots,0}(0,\dots,0,\underbrace{1}_{j-th},0,\dots,0,\underbrace{1}_{k-th},0,\dots,0,\underbrace{1}_{l-th},0,\dots,0,\underbrace{1}_{m-th},0,\dots,0,\underbrace{1}_{n-th},0,\dots,0) =$$

$$a_{111}(1)(I_j + I_k + I_l + I_m) + b_{11111}(1)(I_j + I_k + I_l + I_m + I_n).$$

Again imposing Kirchhoff's law on subfractals and vertices, we get the following equations for **a** and **b**

$$\begin{aligned}
1 + da_4(4) + b_{41}(3) - b_{41}(1) &= 0 \\
(d+1)a_{31}(3) + b_{41}(4) + b_{32}(3) + (d-1)b_{311}(3) - b_{311}(1) &= 0 \\
(d+1)a_{31}(1) + b_{41}(1) + b_{32}(2) + (d-2)b_{311}(1) &= 0 \\
(d+1)a_{22}(1) + b_{32}(3) + b_{32}(2) + (d-1)b_{221}(2) - b_{221}(1) &= 0 \\
(d+1)a_{211}(2) + b_{311}(3) + 2b_{221}(2) + (d-2)b_{2111}(2) - b_{2111}(1) &= 0 \\
(d+1)a_{211}(2) + b_{311}(1) + b_{221}(2) + b_{221}(1) + (d-3)b_{2111}(1) &= 0 \\
(d+1)a_{1111}(2) + b_{2111}(2) + 3b_{2111}(1) + (d-4)b_{11111}(1) &= 0 \\
a_4(4) + a_{31}(3) + 2b_{41}(4) &= 0 \\
a_{31}(1) + 2b_{41}(1) &= 0 \\
a_{31}(3) + a_{22}(2) + 2b_{32}(3) &= 0 \\
a_{31}(1) + a_{22}(2) + 2b_{32}(2) &= 0 \\
2a_{31}(3) + a_{211}(2) + 3b_{311}(3) &= 0 \\
a_{31}(1) + a_{211}(1) + 3b_{311}(1) &= 0 \\
a_{22}(2) + a_{211}(2) + a_{211}(1) + 3b_{221}(2) &= 0 \\
2a_{211}(1) + 3b_{221}(1) &= 0 \\
3a_{211}(2) + a_{1111}(1) + 4b_{2111}(2) &= 0 \\
2a_{211}(1) + a_{1111}(1) + 4b_{2111}(1) &= 0 \\
4a_{1111}(1) + 5b_{11111}(1) &= 0.
\end{aligned}$$

By solving the above equations we can determine inner inward flowing currents corresponding to decimation number $\mathbf{b} = \mathbf{5}$ which appear in section V.

Conclusion

Here in this work it has been rigorously shown that the macroscopic isotropy will be restored if the corresponding renormalization map between two different scales has properties such as: positivity, homogeneity of first order and most of all monotonically increasing property. Obviously, homogeneity is enough and the order of homogeneity does not play a very important role. It is clear that this can be true in many physical phenomena, where we quote only very few of them here: Diffusion in inhomogeneous media [1, 3], elasticity property of rubber or the

network of polymer chains[27], conductivity in random resistor network [4] , flux distribution in josephson junction networks[28]. It would be rather interesting to see whether there exist the restoration isotropy which is not due to positive, homogeneous, and monotonically increasing renormalization map of two different scales.

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